Magnetic moment of an electron gas in crossed, homogeneous electric and magnetic fields

1983 J. Phys. A: Math. Gen. 16 1307

(http://iopscience.iop.org/0305-4470/16/6/024)

J. Phys. A: Math. Gen. **16** (1983) 1307–1313. Printed in Great Britain

Magnetic moment of an electron gas in crossed, homogeneous electric and magnetic fields

P Achuthan, S Benjamin and K Venkatesan

Theoretical Physics Group, Department of Mathematics, Indian Institute of Technology, Madras 600036, India

Received 5 August 1982, in final form 20 October 1982

Abstract. The magnetic properties of a Dirac electron gas in crossed, homogeneous electric ($\boldsymbol{E}$) and magnetic ($\boldsymbol{H}$) fields are studied. An explicit expression for the magnetic moment of the gas is obtained. The calculations in the degeneracy limit of the gas show that (i) for values of $E/H > 0.01$ there is a clear suppression of the transition from paramagnetism to diamagnetism, (ii) there is a weakening of the quasi-periodic oscillations of the magnetic moment observed in the $E = 0$ case, again for $E/H > 0.01$, and (iii) there is a distinct possibility of spontaneous magnetisation for the values of $E/H > 0.9$.

1. Introduction

The magnetic properties of a non-interacting Dirac electron gas in intense magnetic fields near the quantum critical value $H_c = 4.414 \times 10^{13}$ G were studied by Canuto and Chiu (1968). It was observed that, for a given field strength, the magnetisation increases with the particle density to the maximum value and then decreases, eventually becoming negative as higher magnetic states are excited. There was no possibility of spontaneous magnetisation. However, Lee *et al* (1969) suggested a new mechanism called Landau orbital ferromagnetism (LOFER) which leads to spontaneous magnetisation of the gas without the usual spin–spin interactions.

The work of Canuto and Chiu was extended by the present authors (Achuthan *et al* 1982, to be referred to as I) to a particular inhomogeneous magnetic field (IMF) in which the possibility of spontaneous magnetisation was demonstrated as arising purely from the space dependence of the magnetic field, and was not due to any LOFER type mechanism, though this mechanism may exist as well.

It was remarked in I that the energy of the electron in the IMF was not very different from that in the homogeneous case. So it was felt that a search should be made for other IMF in which the electron energy itself will be different from the homogeneous case, and will give appreciable deviations both in magnetic and thermodynamic properties. In this context we shall exploit the similarity of the electron motion in certain IMF and crossed homogeneous electric and magnetic (CHEM) fields.

It has been observed by Landau and Lifshitz (1975) and others (see, e.g., Artsimovich and Lukyanov 1980, Alfvén and Falthammar 1963) that for perturbative inhomogeneities the force acting on the charged particle and also its trajectory will be similar, if such an IMF is replaced by CHEM fields. This was found to be true for another IMF (Achuthan *et al* 1983a) where the inhomogeneity was non-perturbative. Such an analogy seems to hold for a class of two-dimensional magnetic fields that possess neither rotational nor reflection–time-reversal symmetries (Witten 1979). This analogy is further strengthened in the quantum mechanical treatment, where both the CHEM fields and some special types of IMF exhibit the removal of the infinite degeneracy (which existed when there was only a homogeneous magnetic field) with respect to a momentum component in the plane perpendicular to the magnetic field.

In view of the above observations it is natural to expect that the study of the thermodynamic and quantum electrodynamic properties of electrons in CHEM fields is an indirect way to investigate electron properties in IMF.

A further impetus for such a study, independent of the above considerations, is the suggestion that in the magnetosphere of a neutron star there exist not only intense magnetic fields but also electric fields (Daugherty and Lerche 1975).

In this paper we study the problem of the magnetic moment of a relativistic electron gas in homogeneous $\boldsymbol{E}$ and $\boldsymbol{H}$ fields which satisfy the conditions $\boldsymbol{E}\cdot\boldsymbol{H}=0$ and $H^2-E^2>0$. In § 2 we derive the expression for the magnetic moment of the electron gas. In § 3 we present the numerical results of our calculations for the degenerate electron gas. The results show that in addition to other deviations from the $E=0$ case, there is a distinct possibility of spontaneous magnetisation.

2. Magnetic moment

The magnetic moment M is obtained from the thermodynamic potential Ω using the relation $\Omega=-\boldsymbol{M}\cdot\boldsymbol{H}$. For a non-interacting electron gas Ω can be written in terms of the single-particle energy of the electron E_j as (Landau and Lifshitz 1959)

$$\Omega=-kT\sum_j \ln\left(1+\exp\left[(\beta(\tilde{\mu}-E_j)\right]\right) \tag{1}$$

where $\beta=1/kT$ and $\tilde{\mu}$ is the chemical potential plus electron rest energy. The energy of the electron in the CHEM fields is obtained by solving the Dirac equation (Canuto and Chiuderi 1969)†:

$$E_{N,p_x,p_z}=mc^2(1-E^2/H^2)^{1/2}[1+(p_z/mc)^2+2N(H/H_c)(1-E^2/H^2)^{1/2}]^{1/2}+cp_x(E/H) \tag{2}$$

where $N=n+\frac{1}{2}s+\frac{1}{2}$ with $n=0,1,2,\ldots$, describing the Landau levels and $s=\pm 1$ describing the spin polarisations.

The summation in equation (1) indicates a sum over the discrete quantum numbers n and s and integration over the momentum p_z. Furthermore, the summation also implies the inclusion of an operator definition of the density of states, necessitated by the presence of p_x in the energy expression. Using the method developed in I, we get the following expression for Ω:

$$\begin{aligned}\Omega=-\frac{kTV}{2(\pi\lambda_c)^3}\Bigg[&\frac{1}{2}\int_{L(1)}^{U(1)} c(1)\,d\eta\int_{-\infty}^{+\infty}\ln\left(1+\exp[\tilde{\beta}(\mu-\varepsilon_{0,\eta,\zeta})]\right)d\zeta\\&+\sum_{N=1}^{\infty}\left(\int_{L(N+1)}^{U(N+1)} c(N+1)\,d\eta-\int_{L(N)}^{U(N)} c(N)\,d\eta\right)\\&\int_{-\infty}^{+\infty}\ln(1+\exp[\tilde{\beta}(\mu-\varepsilon_{N,\eta,\zeta})])\,d\zeta\Bigg]\end{aligned} \tag{3}$$

† In this reference the fields are chosen such that $\boldsymbol{H}$ is directed along the z axis and $\boldsymbol{E}$ along the y axis with the gauge $A_x=-yH$, $A_y=A_z=0$ and $A_0=-yE$.

where

$$U(N)=(2NH/H_c)^{1/2}+E/H \qquad L(N)=-(2NH/H_c)^{1/2}+E/H$$

$$c(N)=\{(2NH/H_c)-[(p_x/mc)-(E/H)]^2\}^{1/2}$$

$$\tilde{\beta}=mc^2\beta \qquad \mu=\tilde{\mu}/mc^2 \qquad \varepsilon_{N,\eta,\zeta}=E_{N,p_x,p_z}/mc^2$$

$$\eta=p_x/mc \qquad \zeta=p_z/mc.$$

In equation (3) the spin summation has been simplified using the two-fold spin degeneracy of the energy as in the $E=0$ case. Before differentiating Ω with respect to H to obtain the magnetic moment, we transform the limits of the η integrations from $[L, U]$ to $[-1, 1]$, thereby avoiding singularities which would otherwise arise at the limits of the η integrations. Thus we get

$$\begin{aligned}
M=&\frac{kTV}{(\pi\lambda_c)^3}\frac{1}{H_c}\Bigg[\int_{-1}^{+1}(1-t_1^2)^{1/2}\,\mathrm{d}t_1\int_{-\infty}^{+\infty}\ln(1+\exp[\tilde{\beta}(\mu-\varepsilon'_{0,t_1,\zeta})])\,\mathrm{d}\zeta\\
&+\sum_{N=1}^{\infty}\int_{-1}^{+1}(1-t_1^2)^{1/2}\,\mathrm{d}t_1\bigg((N+1)\int_{-\infty}^{+\infty}\ln[1+\exp[\tilde{\beta}(\mu-\varepsilon'_{N,t_1,\zeta})]]\,\mathrm{d}\zeta\\
&-N\int_{-\infty}^{+\infty}\ln(1+\exp[\tilde{\beta}(\mu-\varepsilon_{N,t_1,\zeta})])\,\mathrm{d}\zeta\bigg)\Bigg]\\
&-\frac{Vmc^2}{(\pi\lambda_c)^3}\frac{H}{H_c}\Bigg[\int_{-1}^{+1}(1-t_1^2)^{1/2}\,\mathrm{d}t_1\int_{-\infty}^{+\infty}F'_{0,t_1,\zeta}\frac{\mathrm{d}\varepsilon'_{0,t_1,\zeta}}{\mathrm{d}H}\,\mathrm{d}\zeta\\
&+\sum_{N=1}^{\infty}\int_{-1}^{+1}(1-t_1^2)^{1/2}\,\mathrm{d}t_1\bigg((N+1)\int_{-\infty}^{+\infty}F'_{N,t_1,\zeta}\frac{\mathrm{d}\varepsilon'_{N,t_1,\zeta}}{\mathrm{d}H}\,\mathrm{d}\zeta\\
&-N\int_{-\infty}^{+\infty}F_{N,t_1,\zeta}\frac{\mathrm{d}\varepsilon_{N,t_1,\zeta}}{\mathrm{d}H}\,\mathrm{d}\zeta\bigg)\Bigg]
\end{aligned}\tag{4}$$

where $\varepsilon'_{N,t_1,\zeta}$ and $\varepsilon_{N,t_1,\zeta}$ are obtained from $\varepsilon_{N,\eta,\zeta}$ by substituting

$$\eta=(E/H)+t_1[2(N+1)H/H_c]^{1/2} \qquad \text{and} \qquad \eta=(E/H)+t_1(2NH/H_c)^{1/2}$$

respectively. The F and F' in equation (4) are the Fermi functions.

We know that the electron gas becomes degenerate when $\varepsilon_{N,\eta,\zeta}<\mu$ and $T\to 0$. In this situation, the expression $\ln(1+\exp[\tilde{\beta}(\mu-\varepsilon)])$ appearing in Ω and M can be replaced by $\tilde{\beta}(\mu-\varepsilon)$ and the Fermi functions $F_{N,\eta,\zeta}=F'_{N,\eta m\zeta}=1$. Furthermore, the ζ integrations become finite. With these, the magnetic moment M under the degenerate condition reduces to

$$\begin{aligned}
M=&\frac{1}{\pi^3}\frac{\mu_B V}{\lambda_c^3}\Bigg[\Bigg[\int_{-1}^{+1}\mathrm{d}t_1(1-t_1^2)^{1/2}\int_{-1}^{+1}\mathrm{d}t_2\bigg[d'_0(\mu-\varepsilon'_{0,t_1,t_2})\\
&+H\bigg(\frac{\mathrm{d}d'_0}{\mathrm{d}H}(\mu-\varepsilon'_{0,t_1,t_2})-d'_0\frac{\mathrm{d}\varepsilon'_{0,t_1,t_2}}{\mathrm{d}H}\bigg)\bigg]\\
&+2\sum_{N=1}^{K}\bigg\{(N+1)\int_{-1}^{+1}\mathrm{d}t_1(1-t_1^2)^{1/2}\int_{-1}^{+1}\mathrm{d}t_2\bigg[d'_N(\mu-\varepsilon'_{N,t_1,t_2})\\
&+H\bigg(\frac{\mathrm{d}d'_N}{\mathrm{d}H}(\mu-\varepsilon'_{N,t_1,t_2})-d'_N\frac{\mathrm{d}\varepsilon'_{N,t_1,t_2}}{\mathrm{d}H}\bigg)\bigg]
\end{aligned}$$

$$-N\int_{-1}^{+1} \mathrm{d}t_1(1-t_1^2)^{1/2}\int_{-1}^{+1} \mathrm{d}t_2\Big[d_N(\mu-\varepsilon_{N,t_1,t_2})$$
$$+H\Big(\frac{\mathrm{d}d_N}{\mathrm{d}H}(\mu-\varepsilon_{N,t_1,t_2})-d_N\frac{\mathrm{d}\varepsilon_{N,t_1,t_2}}{\mathrm{d}H}\Big)\Big]\Big\}\Big]\!\Big]. \tag{5}$$

Here again the finite limits of the ζ integration, arising from the degeneracy condition, have been transformed to $[-1, +1]$ by the substitution $\zeta = t_2 d_N$ where

$$d_N = \left(\frac{\{\mu-(E/H)[(E/H)+t_1(2NH/H_c)^{1/2}]\}^2}{1-E^2/H^2}-1-\frac{2NH}{H_c(1-E^2/H^2)^{1/2}}\right)^{1/2}.$$

Further

$$d'_N = \left(\frac{[\![\mu-(E/H)\{(E/H)+t_1[2(N+1)H/H_c]^{1/2}\}]\!]^2}{1-E^2/H^2}-1-\frac{2NH}{H_c(1-E^2/H^2)^{1/2}}\right)^{1/2}$$

and ε_{N,t_1,t_2} and ε'_{N,t_1,t_2} are obtained from $\varepsilon_{N,t_1,\zeta}$ and $\varepsilon'_{N,t_1,\zeta}$ by the replacements $\zeta = t_2 d_N$ and $\zeta = t_2 d'_N$, respectively. The susceptibility χ which is defined through the relation

$$M/V = \chi H \tag{6}$$

can be easily obtained from the expressions for M given in equations (4) and (5).

3. Numerical results and discussion

The magnetisation $\mathcal{M} = M/V$ and susceptibility χ for the degenerate electron gas have been computed using an IBM 370/155, for several values of H/H_c, E/H and $\tilde{\mu}$. Our results for $E = 0$ are identical to those found by Canuto and Chiu (1968). However, for $E \neq 0$, there are marked deviations from the $E = 0$ case. The variations of $\mathcal{M}$ with H/H_c and $\tilde{\mu}$ are shown in figures 1 and 2, respectively, keeping the ratio E/H as a parameter.

Two of the features which were observed, when only a homogeneous magnetic field was present, are suppressed with the increasing values of E/H. These are the quasi-periodic oscillations of χ with variations in $1/H$ (the famous de Haas–van Alphen oscillations, see figure 3) and the paramagnetic-to-diamagnetic transition when higher quantum levels are occupied. With the contribution due to the spin of the electron (paramagnetic part), the orbital motion (diamagnetic part) and also the relativistic effects, all taken together in expressions for $\mathcal{M}$, it is not immediately possible to give a quantitative explanation for the above phenomena. Even at the non-relativistic level, where the separation of the spin and orbital contributions is possible, one has the problem that the density of states factor, in terms of which the oscillatory phenomenon is explained in the $E = 0$ case (Vonsovskii 1974), cannot be obtained explicitly when $E \neq 0$. However, one can give a qualitative explanation as follows. It is known that when only a homogeneous magnetic field is present, the magnetisation is discontinuous if one considers just the transverse (two-dimensional) motion of the electron. The discontinuity is removed in three-dimensional motion where the continuous variable p_z enters the energy spectrum and the magnetisation exhibits a quasi-periodic oscillatory behaviour. With $E \neq 0$, there is another continuous variable, i.e. p_x appearing in the energy expression which may lead to the suppression of oscillations.

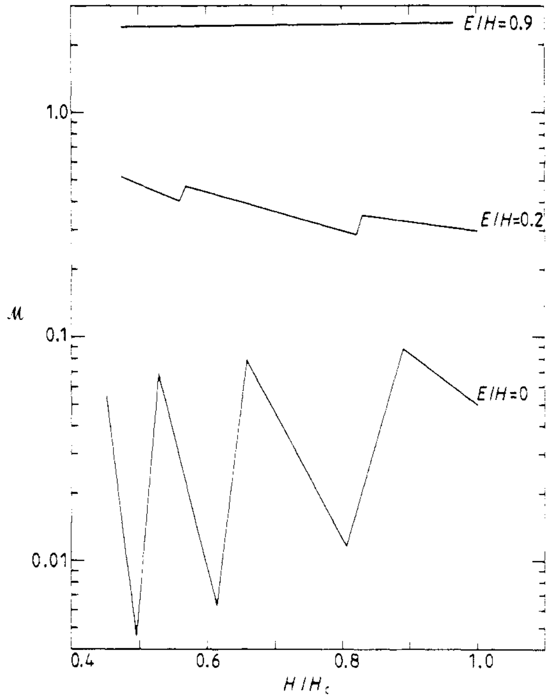

Figure 1. The variation of $\mathcal{M}$ (in units of μ_B/λ_c^3) with H/H_c, for $\mu = 2.5$.

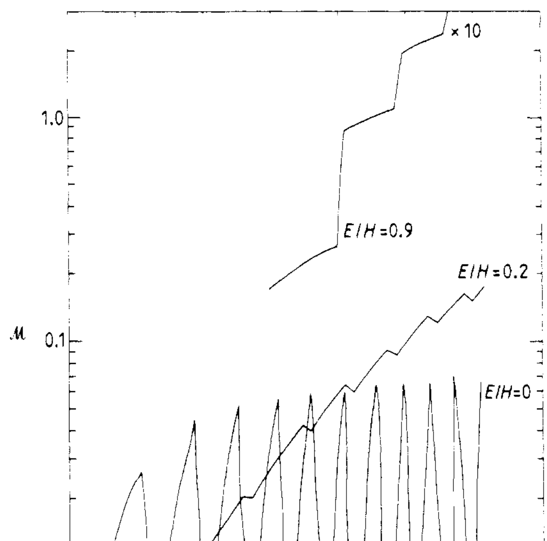

Figure 2. The variation of $\mathcal{M}$ (in units of μ_B/λ_c^3) with μ, for $H/H_c = 0.5$. For $E/H = 0.9$, note that $\mathcal{M}$ is scaled by a factor of ten.

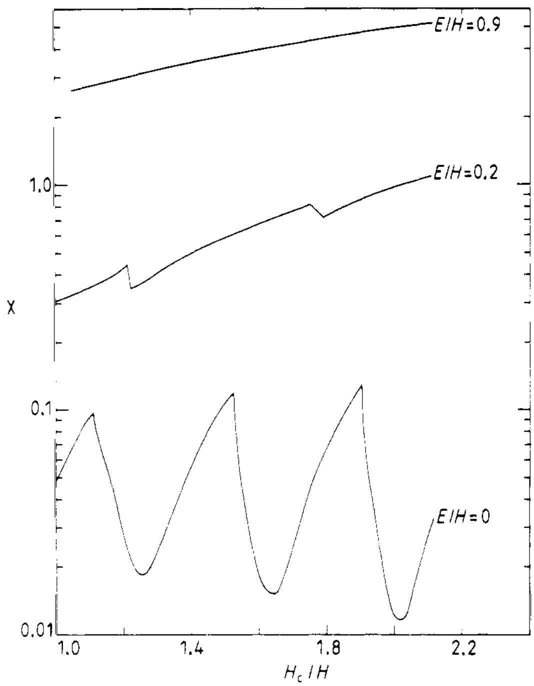

Figure 3. The suppression of the oscillations of the susceptibility χ ($\times 3.65\times 10^{-2}$) with H_c/H for $\mu=2.5$, as E/H is increased.

An interesting observation arising from our calculations is that, for given values of H/H_c and chemical potential μ, the number of levels after which the summation in equation (5) is terminated reduces when E/H is increased. The decrease in the energy of the electron in the highest filled level caused by this would be off-set by the explicit presence of p_x in the energy expression in equation (2), so that the chemical potential is kept at its original value. Now, with the preclusion of the higher Landau levels whose negative contributions make possible the paramagnetic-to-diamagnetic transition, one finds no such transition here for large values of E.

The magnetic induction B inside the electron gas magnetised by the impressed field H is given by

$$B=\mu_0(H+\mathcal{M}) \tag{7}$$

where μ_0 is the magnetic permeability. If the magnetisation $\mathcal{M}$ matches the induction B, the gas may exhibit spontaneous magnetisation. For this to occur one must have

$$\mathcal{M}>H. \tag{8}$$

When $E=0$, it was shown by Canuto and Chiu (1968) that the maximum of $\mathcal{M}$ is only about 10^{-3} H. Thus there was no possibility of spontaneous magnetisation then. Now, with the introduction of the crossed electric field, our results show that the condition (8) can be satisfied for large vaues of E/H. Table 1 lists the values of $\mathcal{M}$ for several choices of E/H corresponding to two values of H/H_c and $\mu=5.0$. It can be seen that for $E/H=0.9$ and $H=2.21\times 10^{13}$ G condition (8) is met. Hence a distinct possibility of spontaneous magnetisation is seen to exist.

Table 1. Change of magnetisation $\mathcal{M}$ with E/H. For $E/H = 0.9$ and $H = 2.21 \times 10^{13}$ G, $\mathcal{M} > H$ which leads to the possibility of spontaneous magnetisation.

	$\mathcal{M}$ (G)	
E/H	$H = 2.21 \times 10^{13}$ G	$H = 4.41 \times 10^{13}$ G
0	2.12×10^{10}	3.41×10^{10}
0.1	4.80×10^{11}	3.08×10^{11}
0.2	1.07×10^{12}	6.03×10^{11}
0.3	1.74×10^{12}	1.02×10^{12}
0.4	2.55×10^{12}	1.53×10^{12}
0.5	3.80×10^{12}	2.12×10^{12}
0.6	5.66×10^{12}	3.14×10^{12}
0.7	8.70×10^{12}	4.95×10^{12}
0.8	1.59×10^{13}	8.67×10^{12}
0.9	4.52×10^{13}	2.29×10^{13}

From the nature of the results of this paper, we can expect that the study of other thermodynamic properties of an electron gas in combined electric and magnetic fields could yield further interesting results. In fact our work on this topic to be published in a companion paper (Achuthan *et al* 1983b) does justify this expectation.

Acknowledgments

One of the authors (SB) is grateful to the Council of Scientific and Industrial Research, New Delhi, for the award of a fellowship. We thank the referees for their valuable suggestions.

References

Achuthan P, Benjamin S and Venkatesan K 1982 *J. Phys. A: Math. Gen.* **15** 3607
——1983a *Phys. Scr.* to be published
——1983b *J. Phys. A: Math. Gen.* **16** to be published
Alfvén H and Falthammar C G 1963 *Cosmical Electrodynamics: Fundamental Principles* (London: Oxford University Press) p 27
Artsimovich L A and Lukyanov S Yu 1980 *Motion of Charged Particles in Electric and Magnetic Fields* (Moscow: Mir) p 105
Canuto V and Chiu H-Y 1968 *Phys. Rev.* **173** 1229
Canuto V and Chiuderi C 1969 *Lett. Nuovo Cimento* **2** 223
Daugherty J K and Lerche I 1975 *Astrophys. Space Sci.* **38** 437
Landau L D and Lifshitz E M 1959 *Statistical Mechanics* (London: Pergamon) p 153
—— 1975 *The Classical Theory of Fields* (Oxford: Pergamon) p 59
Lee H J, Canuto V, Chiu H-Y and Chiuderi C 1969 *Phys. Rev. Lett.* **23** 390
Vonsovskii S V 1974 *Magnetism* vol 1 (New York: Wiley) p 203
Witten E 1979 *Ann. Phys., NY* **120** 72